Celestial mechanics, from a high-school exercise to Newton's Principia

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Abstract

A high-school exercise is used to get an insight into the laws of planetary motion.

1 Introduction: some history

Science contests for high-school students have a hundred-years-old tradition in Hungary [1, 2]. The first one, open to students who just completed their high-school cursus, was organized in 1894 to mark the appointment of L. Eötvös as Minister of Education. This was also the year when the *High-School Journal for Mathematics and Physics* was first published. This Journal sets selected problems each month, and the best solutions are published under the name of the student who solved it.

The early prize-winners of the Eötvös contest include such outstanding future scientists as L. Fejér (who became famous in mathematical analysis), Th. von Kármán (who made important contributions to hydro and aerodynamics), A. Haar (remembered for his invariant measure on group manifolds) . . . followed by many others.

In 1916, in the middle of World War I, a physics contest was held for the first time – and L. Szilárd (who later, with E. Fermi, patented the nuclear reactor) came second. E. Wigner (who got the Nobel prize for using group theory in atomic physics), and J. von Neumann (who built the first computer and contributed to many fields, ranging from game theory to quantum mechanics) only missed the contest because of the turmoil following the disaster of World War I.

In 1925 Edward Teller won the first prize in physics, and has been so impressed by one of the problems of the contest, that he told and retold it during his entire long life [3]. Teller also shared the mathematics prize with Laszlo Tisza. They became friends and their friendship lasted for ever. Soon Tisza (then a mathematics student in Göttingen) changed to physics under the influence of Born, and published his very first paper jointly with Teller (who was preparing his Ph. D. in Leipzig with Heisenberg) on molecular spectrum. Later Tisza introduced the two-fluid model of superfluidity, further developped by Landau.

In the late twenties, von Neumann and Szilárd suggested that the university examinations could be replaced by such a contest [2].

A Fazekasban kezdődött . . .

The tradition still continues, and contributes to forming future generations of scientists [1]. The contests have been internationalized with the *International Olympiads for High-School Students*: first in Mathematics, and, since 1967, also in Physics.

The Eötvös contest lasts 5 hours, and the use of *all* documents or tools is allowed. Three problems that require imagination and creative thinking rather than lexical knowledge are asked. They can, in some cases, lead to genuine research. A famous example is that of A. Császár (a distinguished topologist), who, as a young assistent, was called to survey the contest. While the high-school students were working, Császár figured out a generalization of the geometry problem given that year, and later published a paper on his findings.

2 A problem of spacecraft landing

Some of the physics problems deserve further thinking also. In 1969 – the year of the Apollo Moon landing [1], – for example, an exercise asked the following. *A spacecraft moves on a circular trajectory of radius $r = 4000$ km around a planet of radius $R = 3600$ km. Then, for a short time, the rocket engins (directed oppositely to its motion), are switched on. This puts the spacecraft onto an elliptic trajectory, which touches the planet's surface at the opposite point of the trajectory. Which proportion of its kinetic energy went lost?*

The problem can be solved by elementary tools. It is, however, instructive to describe it also using different, progressively more sophisticated, methods, which provide us with further insight into the intricacies of planetary motion.

From a physical point of view, we have the following situation. As the engine works for a very short time, the position of the spacecraft does not change considerably. Owing to its loss of velocity, the gravitational attraction wins the race and pulls the circular trajectory more tight: the trajectory becomes an ellipse with major axis $2a = r + R$. Our task amounts to comparing the kinetic energies of the two types of motions in the same (aphelion) point.

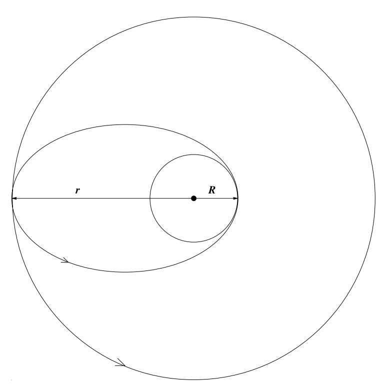

Figure 1: *Reducing the velocity puts the spacecraft onto an elliptic trajectory.*

[1]As the impact speed does not vanish, the problem is obviously *not* a realistic model of Moon landing – but this has not been its aim, neither.

A Fazekasban kezdődött . . .

Belows we establish the following statement: *The ratio of the [aphelion] kinetic energies of the two types of motion is the perihelion-distance divided by the major semi-axis of the elliptical motion,*

$$\delta = \frac{E_a^{kin}}{E_0^{kin}} = \frac{r_p}{a} = \frac{2R}{r+R}. \tag{1}$$

Then the answer to the originally asked question follows at once:

$$\epsilon = \frac{E_0^{kin} - E_a^{kin}}{E_0^{kin}} = 1 - \delta = \frac{r-R}{r+R}. \tag{2}$$

Now, as $e = \frac{1}{2}(R+r) - R = \frac{1}{2}(r-R)$ is the excentricity, (2) is indeed $\epsilon = e/a$: *the relative loss of energy is the numerical excentricity of the new orbit*, i. e., the measure of the “flattening” of the circle. Let us observe that the answer only depends on the geometric dimensions.

With the given numerical data, we find $\epsilon = 1/19$.

Below we present several proofs of (1), ordered following their ever-increasing difficulty and background knowledge.

3 Proofs

Proof I: using Kepler’s laws alone.

The first proof is elementary. According to the laws of circular motion[2],

$$\frac{v_0^2}{r} = \frac{fM}{r^2} \tag{3}$$

where f is Newton’s constant. The [square of the] period is hence

$$T_0^2 = (\frac{2\pi r}{v_0})^2 = 4\pi^2 \frac{r^3}{fM}. \tag{4}$$

This circular motion has kinetic energy

$$E_0^{kin} = \frac{mv_0^2}{2} = \frac{fMm}{2r}. \tag{5}$$

Application of Kepler’s third law to the elliptic motion yields the period of this latter,

$$\frac{T^2}{T_0^2} = \frac{(R+r)^3}{(2r)^3} \quad \Longrightarrow \quad T = \frac{2\pi}{\sqrt{fM}} \left(\frac{R+r}{2}\right)^{3/2}. \tag{6}$$

The area of the ellipse is πab, where b is the minor semi-axis. $b^2 = \sqrt{a^2 - e^2} = \sqrt{rR}$, the area is hence $\pi(R+r)\sqrt{rR}/2$. The areal velocity, which is constant by Kepler’s second law, is therefore

$$\nu = \frac{\pi ab}{T} = \sqrt{\frac{fMrR}{2(R+r)}}. \tag{7}$$

[2] As the answer is obviously independent of the mass of the spacecraft, we chose this latter to be unity.

A Fazekasban kezdődött . . .

At the aphelion $\nu = \frac{1}{2}v_a r$, so that after slowing down, the velocity is

$$v_a = \sqrt{\frac{2fMR}{r(R+r)}}. \tag{8}$$

Then the corresponding kinetic energy is then

$$E_a^{kin} = \frac{mv_a^2}{2} = \frac{fMm}{r}\frac{R}{R+r} = E_0^{kin}\frac{2R}{R+r}, \tag{9}$$

which implies indeed (1).

<u>Proof II: Using the conservation of the energy and of the angular momentum</u>.

Denoting the aphelion and the perihelion velocities by v_a and v_p, the conservation of the energy and of the angular momentum (divided by m) requires that

$$\frac{v_a^2}{2} - \frac{fM}{r} = \frac{v_p^2}{2} - \frac{fM}{R}, \tag{10}$$

$$\frac{v_a \cdot r}{2} = \frac{v_p \cdot R}{2}. \tag{11}$$

Eliminating the perihelion velocity yields once again the kinetic energy (9).

<u>Proof III: Using the formula of the total energy of planetary motion</u>.

An important property of planetary motion [4, 6, 7] is that the total energy only depends on the major axis, according to

$$E^{tot} = -\frac{fMm}{2a}. \tag{12}$$

Then it follows from the energy conservation that, at any point of the trajectory, the velocity satisfies

$$v^2 = fM\left(\frac{2}{r} - \frac{1}{a}\right). \tag{13}$$

For the circular motion, $a = r$, and for the elliptic $a = (r + R)/2$, respectively. Plugging this into (13), yields (5) and (9), respectively.

Even more simply, observing that the change of the total energy is in fact that of the kinetic energy, since the potential energy is unchanged. Using (12) we have

$$\Delta E^{kin} = \Delta E^{tot} = fMm\left(\frac{1}{r+R} - \frac{1}{2r}\right) = (\frac{fMm}{2r})\frac{r-R}{r+R}. \tag{14}$$

Writing here by (5) H_0^{kin} in place of $fM/2r$ yields (2) directly.

It is worth noting that, at a point r of the trajectory, the ratio of the kinetic and the potential energies is, by (13), $E^{kin}/E^{pot} = r/2a - 1$. We have therefore

$$E_0^{kin} = -\tfrac{1}{2}E^{pot}, \qquad \text{resp.} \qquad E_a^{kin} = -\frac{R}{r+R}E^{pot}, \tag{15}$$

which yields again (1).

A Fazekasban kezdődött . . .

Furthermore, while the total energy only depends on the major semi-axis, this is not so for the parts taken individually by the kinetic and the potential energies. According to (15) we have indeed

$$E_0^{kin} = -E_0^{tot}, \qquad E_a^{kin} = -\frac{R}{r} E_a^{tot}. \tag{16}$$

Proof IV: From the radial equation

The result can also be obtained from studying radial motion[3]. For an arbitrary central potential V, the problem can be reduced to one-dimensional motion with effective potential $V_{eff} = V + \frac{L^2}{2mr^2}$ where L is the total angular momentum, cf. [4] p. 75. The radial velocity is therefore given by

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{L^2}{2mr^2}\right)} \tag{17}$$

where $E = E^{tot}$ is the total energy. In the extremal points r_a and r_p the radial velocity vanishes, so that, for $V = -fMm/r$,

$$-\frac{fmM}{r_a} + \frac{L^2}{2mr_a^2} = E - \frac{fmM}{r_p} + \frac{L^2}{2mr_p^2}.$$

Thus

$$\left(\frac{L}{m}\right)^2 = \frac{2}{\dfrac{1}{r_a} + \dfrac{1}{r_p}}\, fM.$$

But the aphelion velocity is $v_a = L/mr_a$ so that

$$v_a^2 = \frac{fM}{r_a^2} \cdot \frac{2}{\dfrac{1}{r_a} + \dfrac{1}{r_p}}. \tag{18}$$

For circular motion at $r = r_a$, $fM/r_a = v_0^2$, and therefore

$$\frac{v_a^2}{v_0^2} = \frac{1}{r_a} \cdot \frac{2}{\dfrac{1}{r_a} + \dfrac{1}{r_p}}, \tag{19}$$

consistently with our previous result.

Proof V: Relation to Kepler's third law.

Kepler's third law is related to the behaviour of the system with respect to scaling [6]: if some trajectory is dilated from the focus by λ, and the time is dilated by $\lambda^{3/2}$,

$$\mathbf{r} \to \mathbf{r}' = \lambda\, \mathbf{r}, \qquad t \to t' = \lambda^{3/2}\, t, \tag{20}$$

yields again a possible trajectory. In those points which correspond to each other, both the kinetic and the potential energies [and hence also the total energy] are related as the inverse ratio of the geometric dimensions,

$$\frac{E'}{E} = \lambda^{-1}. \tag{21}$$

[3] This was suggested by J. Kürti.

A Fazekasban kezdődött . . .

Let us now retract our original circular motion so that its radius equals to the major semi-axis of our elliptic motion above, i.e., consider the dilation by $\lambda = \frac{1}{2}(r+R)/r$. By (21) the total energy [and consistently with (12)] is

$$\tilde{E}_0^{tot} = \frac{2r}{r+R} E_0^{tot}.$$

This is, however, the same as the total energy of the elliptic motion, $\tilde{E}_0^{tot} = E_a^{tot}$, since the major semi-axes are equal. Hence once again

$$E_a^{tot} = \frac{2r}{r+R} E_0^{tot}.$$

Then the result follows from (16).

Let us stress that Kepler's third law did not suffice alone; we also needed the statement about the total energy.

Proof VI: Using the Frenet formulæ.

It is worth describing the motion using the moving local frame introduced by Frénet [8]. Then, for a trajectory of arbitrary shape, the normal component of the acceleration is v^2/ρ where ρ is the radius of curvature i. e., the radius of the osculating circle [8]. In an extremal point of the ellipse the acceleration is normal, and points toward the focus. Hence

$$m\frac{v^2}{\rho} = \text{Force}, \tag{22}$$

which generalizes (3), the formula of circular motion. For the circle $\rho = r$, so that

$$\frac{v_0^2}{r} = \frac{v_a^2}{\rho} \qquad \Longrightarrow \qquad \frac{E_a^{kin}}{E_0^{kin}} = \frac{\rho}{r}, \tag{23}$$

since the force is the same for both problems. We have hence proved: *The ratio of the kinetic energies is identical to that of the radii of curvature.* In the extremal points of the ellipse,

$$\rho = \frac{b^2}{a} = \frac{2rR}{r+R},$$

which implies again (1). This confirms our intuition: decreasing the velocity increases the curvature. Using the explicit form, fMm/r^2, of the force, (22) would allow us to calculate the velocity as

$$\frac{v_a^2}{2} = (\frac{fM}{2r}) \cdot \frac{\rho}{r} = (E_0/m) \cdot \frac{\rho}{r}. \tag{24}$$

This is, however, not necessary for us: it was enough to know the geometric dimensions of the trajectory.

Proof VII: Using the "Runge-Lenz" vector.

A proof analogouos to that in II is obtained if we use the so called "Runge-Lenz" vector [4, 5, 6, 7]

$$\mathbf{K} = m\mathbf{v} \times \mathbf{L} - fMm\,\hat{\mathbf{r}} \tag{25}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{v}$ is the conserved angular momentum; $\hat{\mathbf{r}}$ denotes the unit vector carried by the radius vector drawn form the Earth's center to the spacecraft's position.

A Fazekasban kezdődött . . .

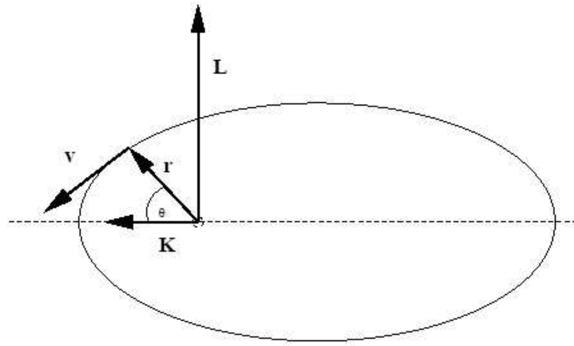

Figure 2: *The conserved Runge-Lenz vector is directed from the Earth's center towards the perihelion position.*

Differentiating **K**-t with respect to time shows, using

$$\dot{\hat{\mathbf{r}}} = \frac{\mathbf{v}}{r} - \frac{(\mathbf{r}\cdot\mathbf{v})}{r^3}\mathbf{r} = \frac{1}{r^3}\left(\mathbf{r}\times(\mathbf{v}\times\mathbf{r})\right) = -\frac{1}{r^3}\mathbf{r}\times(\mathbf{L}/m)$$

and the equation of motion, that **K** is a constant of the motion. The scalar product of **K** with **L** vanishes, so that **K** lies in the plane of the motion; it points from the focus to the perihelion point: $\mathbf{K} = K\hat{\mathbf{e}}$. Multiplying (25) with **r** yields the trajectory [4, 6] as

$$r = \frac{p}{1+\epsilon\cos\theta}, \qquad p = \frac{L^2}{fMm^2} \qquad \epsilon = \frac{K}{fMm} \tag{26}$$

where θ is the angle between **K** and **r**. (26) defines a conic with parameter p and numerical excentricity ϵ.

Returning to our initial problem, let us observe that in the extremal points

$$\mathbf{K} = mv_pL\hat{\mathbf{e}} - fMm\hat{\mathbf{e}} = -mv_aL\hat{\mathbf{e}} + fMm\hat{\mathbf{e}}, \tag{27}$$

where $\hat{\mathbf{e}}$ is the unit vector directed from the center to the perihelion. The length of **L** is clearly $L = mv_pr_p = mv_ar_a$ [cf. (11)]; eliminating the perihelion velocity,

$$\frac{v_a^2}{2} = fM\frac{r_p}{r_a(r_p+r_a)}. \tag{28}$$

For circular motion $r_a = r_p = r$, implying (5); for our elliptic motion $r_p = R$, $r_a = r$ which provides us again with (9), the kinetic energy in the aphelion. Squaring (27) yields furthermore

$$K^2 = f^2M^2m^2 + 2E^{tot}L^2. \tag{29}$$

Hence $K = fMm\epsilon$ which, together with (27) yields

$$v_a^2 = \frac{fM}{r}(1-\epsilon). \tag{30}$$

Writing $2E_0^{kin}$ for fMm/r provides us again with (1) or (2).

A Fazekasban kezdődött . . .

Proof VIII. Using the hodograph

Drawing the instantaneous velocity vector from a fixed point O of "velocity space" yields the *hodograph*. For planetary motion this is a circle [4, 7]. The simplest proof of this statement is obtained if we multiply the angular momentum vectorially with the Runge-Lenz vector [4]. Developping the double vector product and using $\mathbf{L}\cdot\mathbf{v}=0$ yields

$$\mathbf{L}\times\mathbf{K}=L^2m\mathbf{v}-fMmL\mathbf{u},$$

where $\mathbf{u}$ is the unit vector obtained by rotating $\hat{\mathbf{r}}$ around the direction of $\mathbf{L}$ by 90^o degrees in the counter-clockwise direction. As $\mathbf{L}\times\mathbf{K}=-LK\mathbf{j}$ where $\mathbf{j}$ is the unit vector directed along the y axis of the coordinate plane. Writing here $K=fMm\epsilon$ the velocity vector is expressed as

$$\mathbf{v}=\frac{fM}{L}\Big(\mathbf{u}-\epsilon\mathbf{j}\Big). \tag{31}$$

As the unit vector $\hat{\mathbf{r}}$ turns around during the motion, so does also $\mathbf{u}$ (advanced with (90^o degrees). The first term in (31) describes hence a circle of radius fM/L, whose center has been translated to C, situated on the y axis at distance $-K/L=fM\epsilon/L$ below O. For a circular trajectory $C=O$ and the hodograph becomes a circle around the origin with radius $v_0^2=fM/r$.

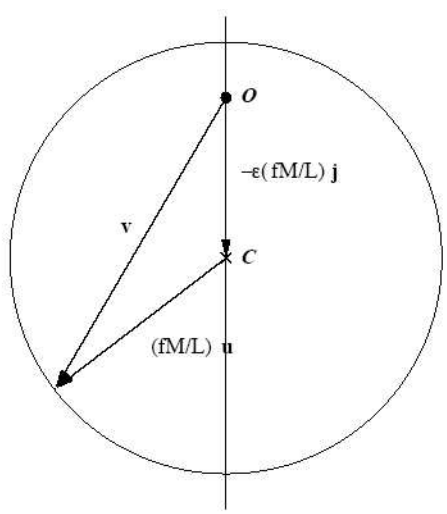

Figure 3: *The velocity vector of planetary motion describes a circle of radius fM/L and center at C, situated on the y axis at distance ($\overline{OC}=K/L=\epsilon fM/L$) in "velocity space". The aphelion point corresponds to the shortest distance from O, represented as the top of the circle.*

The velocity is the largest at the bottom of the circle, which corresponds to the perihelion point. The smallest velocity is obtained in turn in the aphelion, which is the top of the circle. Then $\mathbf{u}$ points vertically upwards, $\mathbf{u}_a=\mathbf{j}$. Then the length of this smallest velocity is plainly

$$\text{smallest velocity is} \quad = \quad \text{(radius)} - \text{(distance } \overline{OC}\text{)},$$

which yields (30). Alternatively, we can write $L=v_ar_a$ in (31).

A Fazekasban kezdődött . . .

Proof IX. Maxwell's generalization

It is worth noting that our problem can be generalized. J. C. Maxwell [9] has in fact shown that, at an *arbitrary* point r of the trajectory, the velocity is

$$v^2 = \frac{4\pi^2 a^2}{T^2}\left(\frac{2a}{r} - 1\right). \tag{32}$$

Maxwell's proof only uses Kepler's second law and some geometric properties of the ellipses. Let in fact the Sun be in one of the foci, S, let H be the other focus and draw perpendiculars SY and HZ from the foci to the tangent to the ellipse at a point P, see Fig. 4.

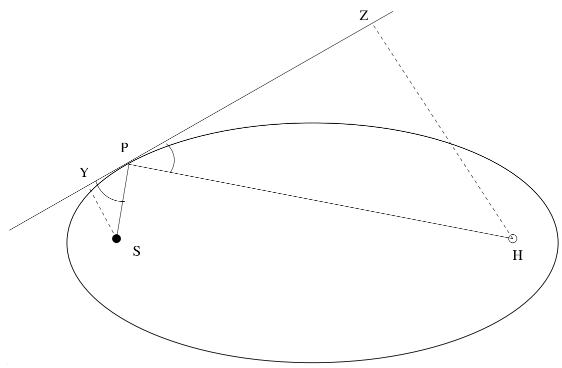

Figure 4: *Maxwell's geometric derivation.*

By Kepler's second law, the areal velocity is constant and given by twice the area of the ellipse divided by the revolution time, T,

$$\overline{SY}\cdot v = \frac{2\pi ab}{T} \qquad \Longrightarrow \qquad v = \frac{2\pi ab}{T}\frac{1}{\overline{SY}}, \tag{33}$$

On the other hand, a known geometric property of the ellipse says that the half minor axis, b, is the mean of $\overline{SY}$ and $\overline{HZ}$,

$$\overline{SY}\cdot\overline{HZ} = b^2.$$

Finally, the "optical property" [known to Kepler] says that all light rays emitted from one focus are reflected by the ellipse to the other focus. It follows that the angles HPZ and SPY are equal; the triangles SPY and HPZ are therefore similar. Hence

$$\overline{SP} : \overline{PH} = \overline{SY} : \overline{ZH}.$$

Multiplication of the two relations yields

$$\overline{SY}^2 = b^2\frac{\overline{SP}}{\overline{PH}} = b^2\frac{r}{2a - r},$$

since $\overline{SP} + \overline{PH} = 2a$, the major axis of an ellipse. Inserting $\overline{SY}^2$ into the square of (33) yields Maxwell's formula (32).

A Fazekasban kezdődött . . .

For a circular orbit in particular, $v_0^2 = 4\pi^2 r^2/T_0^2$. (In general, i. e. with the exception of the extremal points, the velocities $\mathbf{v}_0$ and $\mathbf{v}$ have different directions.) Hence

$$\frac{v^2}{v_0^2} = \frac{a^2}{r^2}\frac{T_0^2}{T^2}\frac{2a-r}{r}.$$

Using here Kepler's III law, we end up with the following simple expression:

$$\delta = \frac{v^2}{v_0^2} = \frac{2a-r}{a}. \tag{34}$$

In words: *At any point of the orbit, the square of the velocity to that of the circular motion through the same point is as the distance from the other focus to the semi major axis*! For $r = r_a$ we recover in particular our previous result (1).

Proof X: From Newton's Principia

Last but not least, we point out that the problem discussed here has actually been solved by Newton as much as 300 years ago! Proposition 16. Theorem 8. Corollary 3. of his "Principia" [10] says in fact that "*the velocity in a conic, at the greatest or least distance from the focus, is to the velocity with which the body would move in a circle, at the same distance from the center, as the square root of the principal latus rectum is to the square root of twice that distance.*"

Translated into our notations,

$$v_a^2 : v_0^2 = \ell : 2r. \tag{35}$$

But the half of the "principal latus rectum" is the harmonic mean of the periapsis and the apoapsis of the ellipse [11] (and hence also the parameter $p = b^2/a$),

$$\frac{2}{\ell} = \frac{1}{2}\left(\frac{1}{r_p} + \frac{1}{r_a}\right) \quad\Longrightarrow\quad \ell = \frac{4rR}{r+R}.$$

Inserting into (35) the result (1) is recovered once again.

Newton's original proof comes by a series of elementary geometric demonstrations using hardly more then properties of similar triangles [10]. The reader is referred to Newton's Principia for details. Let us remark, though, that (34) provides a quick proof to Newton's statement (35), since the prodcuct of $r_a = a(1+\epsilon)$ and $r_p = a(1-\epsilon)$ is $r_a r_p = a^2(1-\epsilon^2) = a\ell/2$.

4 Discussion

Let us review our various approaches. Our first proof only used Kepler's laws specific for the planetary motion, and suits perfectly to a high-school student. The second and seventh proof is based on conservation laws; the second uses that of the energy and the angular momentum, and the last the Runge-Lenz vector. This is early 19th century physics: the vector (25) was in fact introduced by Laplace in 1799, in his *Traité de Mécanique Céleste* [4, 5, 7].

Proof IV is based on the radial equation.

A Fazekasban kezdődött . . .

II, using high-school knowledge only, would clearly work for any conservative force, while VII is related to the "hidden" symmetry of the Kepler problem. Although some of the results could be obtained by freshmen, this approach is not in general taught at the university. It leads to a group theoretical treatment of planetary motion [7]. For example, $\mathbf{L} \cdot \mathbf{K} = 0$ and (29) are the classical counterparts of the Casimir relations of the SO(4) dynamical symmetry used by Pauli to determine the spectrum of the hydrogen atom [7].

III and V are based on the formula (12) of the total energy, discussed by university textbooks [4, 6]. V is linked to the scaling property which yields in fact Kepler's third law [6].

Proof V uses the general framework of co-moving coordinates called the Frénet formulæ [8] (late 19th century), which makes part of regular university courses on mechanics and/or differential geometry. It can be applied to any central force problem: the reader is invited to work out what happens, e. g., for a harmonic force $\mathbf{F} = -k\mathbf{r}$ (when the trajectories are again ellipses.)

Proof VIII is based on the hodograph.

The problem is readily generalized following Maxwell [9] (Proof IX), along the lines of Proof I and adding some knowledge of geometry. Remarkably, this general answer might have been found by Kepler in the early 17th century!

Our final Proof X uses a statement made by Newton in his Principia, also obtained by elementary geometry.

Interestingly, the first problem of the very first Eötvös physics contest (that of Szilard in 1916), was on planetary motion: *Let us consider a miniaturized model of the solar system, where all distances are scaled as* $1 : 15 \cdot 10^{10}$. *How long would last a year?* The answer is that it would remain a year, since, assuming that all densities remain constant, the Sun's mass would also decrease as the cube of its dimension. But this is precisely the condition, as seen from the scaling argument of Proof V [6].

A Fazekasban kezdődött . . .

References

[1] M. Vermes, Eötvös contests 1916-1958. (in Hungarian). Tankönyvkiadó: Budapest (1965); M. Vermes, Eötvös contests 1959-1988. (in Hungarian). Nemzeti Tankönyvkiadó, Typotex: Budapest (1997).
G. Radnai, Eötvös contests 1988-1997. (in Hungarian). Nemzeti Tankönyvkiadó, Typotex: Budapest (1998). On the Eötvös contest in English: see G. Radnai's article in *Hungarian High-School Mathematics and Physics Journal* No. 11, p. 10-15 (1994). For a selection of Hungarian high-school problems, see P. Gnädig, G. Honyek, and K. Riley, *200 Puzzling Physics Problems* (*with hints and solutions*). Cambridge University Press (2001).

[2] G. Radnai and R. Kunfalvi, *Physics in Budapest. A survey.* Amsterdam: North Holland (1988).

[3] E. Teller with J. Shoolery, Memoirs. A twentieth-century journey in science and politics. Perseus Publ. (2001)

[4] H. Goldstein, *Classical Mechanics.* Addison-Wesley: Reading (1980).

[5] H. Goldstein, Am. Journ. Phys. **43**, 735 (1975) and **44** 1123 (1976).

[6] L. Landau, J. Lifchitz: *Mécanique.* Editions MIR: Moscou (1969).

[7] B. Cordani, *The Kepler problem.* Birkhäuser: Basel (2003).

[8] A. Grey, *Modern Differential Geometry of Curves and Surfaces.* Studies in Advanced Mathematics. CRC Press: Ann Arbor (1993).

[9] J. C. Maxwell, *Matter and motion* (1877). Reprinted by Dover (1991).

[10] Sir Isaac Newton, *Philosophiae Naturalis Principia Mathematica. The Principia. Mathematical Principles of Natural Philosophy.* A new translation by I. B. Cohen and A. Whitman, assisted by J. Budenz. p. 469. Univ. Calif. Press, Berkeley (1999).

[11] H. S. M. Coxeter *Introduction to Geometry.* 2nd ed. New York: Wiley, pp. 116-118 (1969).

A Fazekasban kezdődött . . .